MODULE-II

The general form of a second order linear DE with constant co-efficient is given by,

$$y'' + ay' + by = f(x) \quad \text{——— (1)}$$

If $f(x) = 0$ then equation (1) is called homogeneous

If $f(x) \neq 0$ then equation (1) is called non homogeneous

Let us consider a second order linear homogeneous DE with constant co-efficient i.e

$$y'' + ay' + by = 0 \quad \text{——— (2)}$$

$$\Rightarrow \frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

$$\Rightarrow (D^2 + aD + b)y = 0 \qquad \left(\because D = \frac{d}{dx}\right)$$

The auxilary equation or characterstics equ^n is

$$\lambda^2 + a\lambda + b = 0$$

which is a 2nd degree equation in λ.

So it has atmost two solutions or roots.

The relation between the roots are

(i) Real unequal

(ii) Real equal

(iii) Complex and conjugate

CASE-I (Real unequal)

Let the roots be λ_1, λ_2 $(\lambda_1 \neq \lambda_2)$

The general solution is

$$\boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$$

Ex:- $-2, 3$

g.s $y = c_1 e^{-2x} + c_2 e^{3x}$

$x^2 - 3x + 2 = 0$

$\Rightarrow (x-1)(x-2) = 0$

$\Rightarrow x = 1, 2$

$x^2 + 2x + 1 = 0$

$\Rightarrow (x+1)^2 = 0$

$\Rightarrow x = -1, -1$

$x^2 + 2x + 2 = 0$

$\Rightarrow x = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$

$= \frac{-2 \pm \sqrt{-4}}{2}$

$= \frac{-2 \pm 2i}{2}$

$= -1 \pm i$

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CASE-II (Real equal)

Let the roots be λ, λ.

The general solution is

$$y = (C_1 + C_2 x) e^{\lambda x}$$

Ex:- 4, 4

g.s $y = (C_1 + C_2 x) e^{4x}$

CASE-III (complex and conjugate)

Let the roots be $\alpha \pm i\beta$

The general solution is

$$y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

Ex:- $-2 \pm 3i$

The g.s $y = e^{-2x} (C_1 \cos 3x + C_2 \sin 3x)$

Example

Let the roots be $-3, 2, 4, 4, -4 \pm 5i$

The general solution is

$$y = C_1 e^{-3x} + C_2 e^{2x} + (C_3 + C_4 x) e^{4x} + e^{-4x} (C_5 \cos 5x + C_6 \sin 5x)$$

STEPS FOR SOLVING SECOND ORDER LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

STEP:-I Find out the auxilary or the characterstics equation.

STEP:-II Find out the roots by middle term factor or by using the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

STEP:-III see the roots belongs to case-I, case-II or case-III

STEP:-IV Apply the formula accordingly.

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NO-1-TO-9

Find a general solution

9. $y'' + 2Ky' + K^2y = 0$

$\Rightarrow \frac{d^2y}{dx^2} + 2K\frac{dy}{dx} + K^2y = 0$

$\Rightarrow (D^2 + 2KD + K^2)y = 0 \qquad (\because D = \frac{d}{dx})$

The auxilary equation is

$\lambda^2 + 2K\lambda + K^2 = 0$

$\Rightarrow (\lambda + K)^2 = 0$

$\Rightarrow \lambda = -K, -K$

The general solution is

$y = (C_1 + C_2 x)e^{-Kx}$

NO-10-TO-17

Solve the following initial value problems

10. $y'' + y' - 6y = 0 \qquad y(0) = 10 \quad y'(0) = 0$ ——— (1)

$\Rightarrow \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

$\Rightarrow (D^2 + D - 6)y = 0 \qquad (\because D = \frac{d}{dx})$

The auxilary equation is

$\lambda^2 + \lambda - 6 = 0$

$\Rightarrow (\lambda + 3)(\lambda - 2) = 0$

$\Rightarrow \lambda = -3, 2$

So, the general solution is

$y = C_1e^{-3x} + C_2e^{2x}$ ——— (2)

$y' = -3C_1e^{-3x} + 2C_2e^{2x}$ ——— (3)

$y(0) = 10$ i.e $y = 10$ when $x = 0$

$C_1 + C_2 = 10$ ——— (4)

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$y'(0) = 0$ i.e $y' = 0$ when $x = 0$ from (3)

$$-3c_1 + 2c_2 = 0 \quad \text{——— (5)}$$

Solving equation (4) and (5)

$$c_1 = 4, \quad c_2 = 6$$

Putting the value of c_1 and c_2 in equation (2) the solution is

$$y = 4e^{-3x} + 6e^{2x}.$$

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5-TO-12

Find a general solution.

8. $y'' + 4y' + (4+\omega^2)y = 0$

$$\Rightarrow \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + (4+\omega^2)y = 0$$

$$\Rightarrow [D^2 + 4D + (4+\omega^2)]\, y = 0 \qquad \left(\because D = \frac{d}{dx}\right)$$

The auxilary equation is

$$\lambda^2 + 4\lambda + (4+\omega^2) = 0$$

$$\Rightarrow \lambda = \frac{-4 \pm \sqrt{16 - 4\cdot 1\cdot (4+\omega^2)}}{2\cdot 1}$$

$$= \frac{-4 \pm \sqrt{16 - 16 - 4\omega^2}}{2}$$

$$= \frac{-4 \pm \sqrt{4\omega^2 i^2}}{2} = \frac{-4 \pm 2\omega i}{2} = -2 \pm \omega i$$

Thus, the general solution is

$$y = e^{-2x}(c_1 \cos \omega x + c_2 \sin \omega x)$$

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13-TO-18

Solve the following initial value problems

13. $9y'' + 6y' + y = 0$ ——(1) $\quad y(0) = 4, \quad y'(0) = -\frac{13}{3}$.

$\Rightarrow 9\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + y = 0$

$\Rightarrow (9D^2 + 6D + 1)y = 0 \qquad \left(D = \frac{d}{dx}\right)$

The auxilary equation is

$9\lambda^2 + 6\lambda + 1 = 0$

$\Rightarrow (3\lambda + 1)^2 = 0$

$\Rightarrow \lambda = -\frac{1}{3}, -\frac{1}{3}$.

The general solution is

$y = (c_1 + c_2 x)\, e^{-\frac{1}{3}x}$ ——(2)

$y' = c_2 e^{-\frac{1}{3}x} - \frac{1}{3} e^{-\frac{1}{3}x} (c_1 + c_2 x)$ ——(3)

$y(0) = 4$ i.e $y = 4$ when $x = 0$ from (2)

$c_1 = 4$ ——(4)

$y'(0) = -\frac{13}{3}$ i.e $y' = -\frac{13}{3}$ when $x = 0$ from (3)

$c_2 - \frac{1}{3} c_1 = -\frac{13}{3}$ ——(5)

Solving equation (4) and (5)

$c_1 = 4, \quad c_2 = -3$

Putting the value of c_1 and c_2 in equation (2) the solution is

$y = (4 - 3x)\, e^{-\frac{1}{3}x}$.

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19-To-22

Solve the following boundary value problems.

21. $y'' + 2y' + 2y = 0$ $\quad y(0) = 1 \quad y(\frac{\pi}{2}) = 0$ ——(1)

$\Rightarrow \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

$\Rightarrow (D^2 + 2D + 2)y = 0 \qquad (D = \frac{d}{dx})$

The auxilary equation is

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

The general solution is

$$y = e^{-x}(C_1 \cos x + C_2 \sin x) \quad \text{——} (2)$$

$y(0) = 1$ i.e $y = 1$ when $x = 0$ from (2)

$$1 = e^{-0}(C_1 \cos 0 + C_2 \sin 0)$$

$$\Rightarrow 1 = C_1 \quad \text{——} (3)$$

$y(\frac{\pi}{2}) = 0$ i.e $y = 0$ when $x = \frac{\pi}{2}$ from (2)

$$0 = e^{-\frac{\pi}{2}}(C_1 \cos \tfrac{\pi}{2} + C_2 \sin \tfrac{\pi}{2})$$

$$\Rightarrow C_1 \cos \tfrac{\pi}{2} + C_2 \sin \tfrac{\pi}{2} = 0$$

$$\Rightarrow C_2 = 0 \quad \text{——} (4)$$

Putting the value of C_1 and C_2 in equation (2) the solution is

$$y = e^{-x} \cos x.$$

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5-TO-13

5. $(D^2-D-2)y=0$

The auxilary equation is

$$\lambda^2-\lambda-2=0$$

$$\Rightarrow \lambda^2-2\lambda+\lambda-2=0$$

$$\Rightarrow \lambda(\lambda-2)+1(\lambda-2)=0$$

$$\Rightarrow (\lambda+1)(\lambda-2)=0$$

$$\Rightarrow \lambda=-1,2$$

So, the general solution is

$$y=c_1e^{-x}+c_2e^{2x}$$

11. $(64D^2+16D+1)y=0$

The auxilary equation is

$$64\lambda^2+16\lambda+1=0$$

$$\Rightarrow (8\lambda+1)^2=0$$

$$\Rightarrow (8\lambda+1)(8\lambda+1)=0$$

$$\Rightarrow \lambda=-\frac{1}{8},-\frac{1}{8}.$$

So, the general solution is

$$y=(c_1+c_2x)e^{-\frac{1}{8}x}.$$

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